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Linearization of Turbulent Boundary-Layer Equations

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Introduction

IT is common practice to linearize the boundary-layer equations with Newton's method, which usually provides quadratic convergence. The linearization of the laminar-flow equations is straightforward but the expressions used to model the Reynolds stresses present difficulties when it is applied to the turbulent flow equations. Indeed, in previous solutions of the turbulent boundary-layer equations that use the eddy-viscosity concept in algebraic form, Newton's method has been applied to all terms in the equation except for the viscous term where values of the eddy viscosity were assumed from a previous iteration. In this way the linearized system of equations has been solved with prior knowledge of the terms that represent turbulent diffusion and with the consequence that solutions oscillate and the convergence rate is slower than it need be. In a typical airfoil calculation, about four iterations are required at each longitudinal station to achieve a 1% tolerance error in wall shear stress; with, say, 50 stations the additional calculation effort is clearly significant.

The calculation times that result from this incomplete application of Newton's method in the solution of the turbulent boundary-layer equations may be tolerated in situations where the freestream boundary condition is prescribed. However, where solutions of the inviscid- and viscous-flow equations are required to interact, several sweeps of the flowfield may be needed and the additional computer time becomes important. The method described and evaluated here was devised to allow the application of Newton's method to all the terms in the momentum equation. The approach makes use of the eddy-viscosity formulation of Cebeci and Smith[1] and the Mechul function of Ref. 2, which regards the dimensionless displacement thickness and wall shear of the eddy-viscosity formula as unknowns.

Basic Equations

The boundary-layer equations and their boundary conditions for two-dimensional incompressible laminar and turbulent flows are well known. With the concept of eddy viscosity ϵ_m, and with $b = 1 + \epsilon_m/\nu$, they can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_e\frac{\mathrm{d}u_e}{\mathrm{d}x} + \nu\frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial y}\right) \qquad (2)$$

$$y=0 \quad u=v=0; \qquad y\to\delta \quad u\to u_e \qquad (3)$$

The presence of the eddy viscosity ϵ_m in b requires a turbulence model, and the algebraic eddy-viscosity formulation of Cebeci and Smith is used here. According to this formulation, ϵ_m is defined by two separate formulas given by

$$\epsilon_m = \left\{0.4y\left[1-\exp\left(\frac{-y}{A}\right)\right]\right\}^2\frac{\partial u}{\partial y} \quad 0\le y\le y_c \qquad (4a)$$

$$= 0.0168\int_0^\infty (u_e - u)\,\mathrm{d}y \quad y_c\le y\le\delta \qquad (4b)$$

where

$$A = 26\nu u_\tau^{-1}N, \quad N = (1-11.8p^+)^{-1/2},$$

$$u_\tau = \left(\frac{\tau_w}{\rho}\right)^{1/2}, \quad p^+ = \frac{\nu u_e}{u_\tau^3}\frac{\mathrm{d}u_e}{\mathrm{d}x} \qquad (5)$$

The condition used to compute y_c is the continuity of the eddy viscosity; from the wall outward (inner region) Eq. (4a) is applied until its value is equal to the one given for the outer region by Eq. (4b).

For external flows, it is convenient to solve Eqs. (1) and (2) when they are expressed in transformed variables, and the Falkner-Skan transformation is used for this purpose. The result, with $m = x/u_e\, \mathrm{d}u_e/\mathrm{d}x$, is

$$(bf'')' + \frac{m+1}{2}ff'' + m[1-(f')^2] = x\left(f'\frac{\partial f'}{\partial x} - f''\frac{\partial f}{\partial x}\right) \qquad (6)$$

$$\eta=0, \quad f'=f=0; \qquad \eta\to\eta_e, \quad f'\to 1 \qquad (7)$$

In terms of transformed variables, the b term in Eq. (6) can be written as

$$b = 1 + a_1 f''[1-\exp(-a_2N^{-1}f_w''^{1/2})]^2\lambda_1 + a_3(\eta_e - f_e)\lambda_2 \qquad (8)$$

Here λ_1 and λ_2 are determined by the continuity of eddy-viscosity formulas with $\lambda_1 = 1$ and $\lambda_2 = 0$ in the inner region and $\lambda_1 = 0$ and $\lambda_2 = 1$ in the outer region. a_1, a_2, a_3, and N are defined by

$$a_1 = 0.16R_x^{1/2}\eta^2, \quad a_2 = (R_x^{1/2}/26)\eta, \quad a_3 = 0.0168R_x^{1/2}$$

$$a_4 = 11.8mR_x^{-1/4}, \quad N = [1-a_4(f_w'')^{-3/2}]^{-1/2}, \quad R_x = u_e x/\nu \qquad (9)$$

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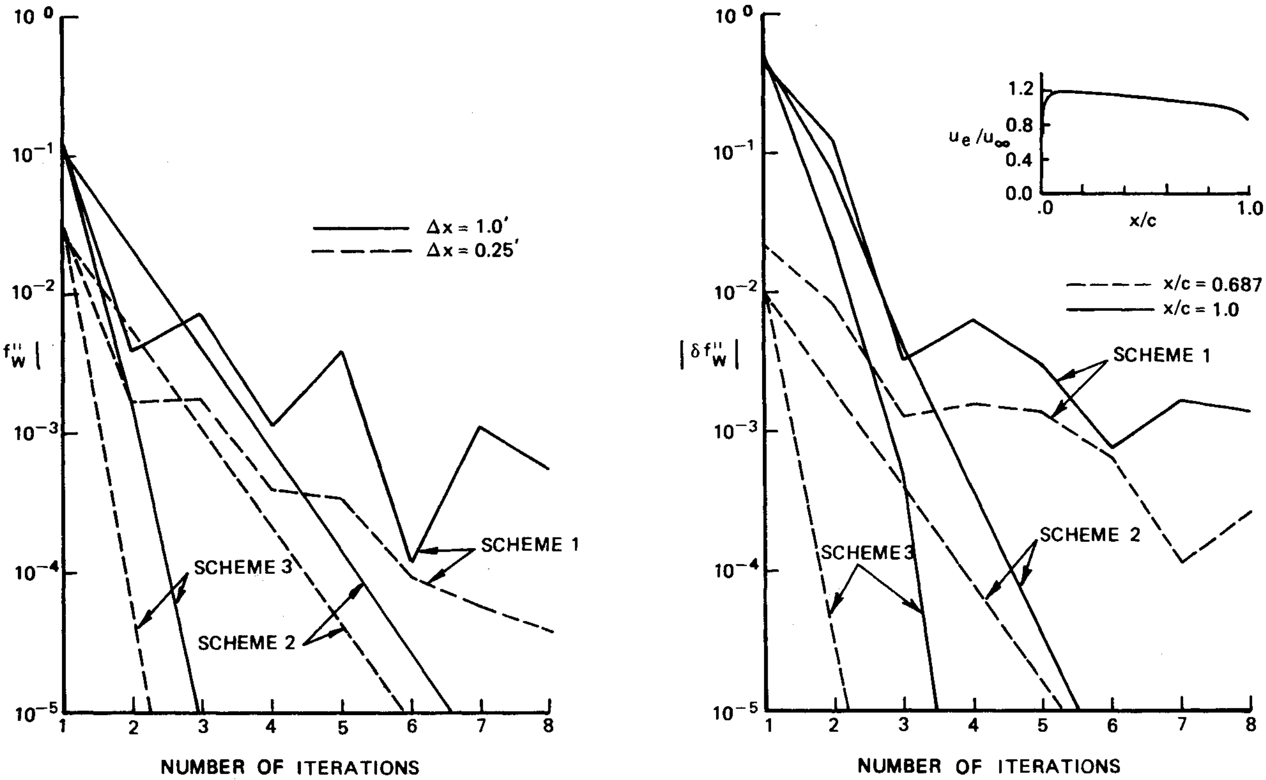

Fig. 1 Rate of convergence of three linearization schemes for a flow with a) zero pressure gradient and b) pressure gradient.

Solution Procedure

The solution of Eqs. (6) and (7) can be obtained by using a finite difference method such as those due to Crank-Nicolson[3] and Keller.[4] As discussed in Ref. 5, the latter has a number of very desirable features, especially for turbulent flows, and is used here. The description of this method for turbulent boundary layers is discussed in several references, e.g., Ref. 5, and only an outline is presented below.

Introduction of new variables u and v, defined by

$$f' = u, \qquad u' = v \tag{10a}$$

allows Eq. (6) to be written as a first-order equation

$$(bv)' + \frac{m+1}{2} fv + m(1-u^2) = x\left(u\frac{\partial u}{\partial x} - v\frac{\partial f}{\partial x}\right) \tag{10b}$$

Next, on a finite difference net denoted by

$$x_0 = 0, \quad x_n = x_{n-1} + k_n, \quad n = 1, 2, \ldots, N$$

$$\eta_0 = 0, \quad \eta_j = \eta_{j-1} + h_j, \quad j = 1, 2, \ldots, J; \quad \eta_J = \eta_e \tag{11}$$

we approximate the quantities (f,u,v) at points (x_n, η_j) by (f_j^n, u_j^n, v_j^n). The finite difference approximation for Eqs. (10a) are then written using centered difference quotients and averaged about the midpoint $(x_n, \eta_{j-1/2})$ and those for Eq. (10b) are written at the middle of the net $(x_{n-1/2}, \eta_{j-1/2})$; this gives

$$f_j^n - f_{j-1}^n - (h_j/2)(u_j^n + u_{j-1}^n) = 0 \tag{12a}$$

$$u_j^n - u_{j-1}^n - (h_j/2)(v_j^n + v_{j-1}^n) = 0 \tag{12b}$$

$$h_j^{-1}[(bv)_j^n - (bv)_{j-1}^n] + \alpha_1 (fv)_{j-1/2}^n - \alpha_2 (u^2)_{j-1/2}^n + \alpha(v_{j-1/2}^{n-1} f_{j-1/2}^n - f_{j-1/2}^{n-1} v_{j-1/2}^n) = R_{j-1/2}^{n-1} \tag{12c}$$

where

$$R_{j-1/2}^{n-1} = -L_{j-1/2}^{n-1} + \alpha[(fv)_{j-1/2}^{n-1} - (u^2)_{j-1/2}^{n-1}] - m^n$$

$$L_{j-1/2}^{n-1} = \{h_j^{-1}(b_j v_j - b_{j-1} v_{j-1}) + m_1 (fv)_{j-1/2} + m[1 - (u^2)_{j-1/2}]\}^{n-1}$$

$$\alpha_1 = \alpha + m_1, \quad \alpha_2 = \alpha + m, \quad \alpha = (x_{n-1/2}/k_n), \quad m_1 = (m+1)/2$$

The next step is to linearize the system just given with Newton's method. The Mechul-function formulation of Ref. 2 is used for this purpose and results in an increase in the number of first-order equations from three to five. For this purpose we let

$$w = f_e, \qquad s = f''_w \tag{13}$$

so that the b term in Eq. (8) can be written as

$$b_j = 1 + a_1 v_j [1 - \exp(-a_2 N^{-1} s_j^{1/2})]^2 \lambda_1 + a_3(\eta_J - w_j)\lambda_2 \tag{14}$$

where now $N = [1 - a_4 (s_j)^{-3/2}]^{-1/2}$. Differentiating Eqs. (13) with respect to η, we get

$$w' = 0, \qquad s' = 0 \tag{15}$$

The difference equations for Eqs. (15) are

$$w_j - w_{j-1} = 0, \qquad s_j - s_{j-1} = 0 \tag{16}$$

The boundary conditions given by Eq. (7) become

$$f_0 = u_0 = 0, \quad s_0 = v_0, \quad u_J = 1, \quad w_J = f_J \tag{17}$$

The system given by Eqs. (12), (16), and (17) with b_j given by Eq. (14) inserted in Eq. (12c) can now be linearized by using Newton's method and the resulting linear system for the five unknowns, f, u, v, w, and s can be solved by the block-elimination method discussed in Ref. 5.

Results and Discussion

Results are obtained for two separate flows consisting of a zero pressure gradient and an adverse pressure gradient. Calculations are done for three different linearization schemes. The first one is the usual procedure in which the b term in Eq. (12c) is assumed to be known from a previous iteration, the second one is to neglect the variation of f_w'' in the eddy-viscosity formulas ($s \equiv 0$) and assume it to be known from a previous iteration. In the latter case we have four unknowns (f, u, v, and w), in contrast to the third linearization scheme in which we have five unknowns (f, u, v, w, and s), as described in the previous section, and all the terms in the equations are linearized fully by Newton's method.

Figure 1a shows the rate of convergence of the three linearization schemes for the case of a zero pressure-gradient flow. The calculations were started as laminar for a unit Reynolds number of 10^6/ft at the leading edge of the plate; the transition was specified very close to the leading edge and approximately eight x stations were taken in the region $0<x<1$. Between $x=1$ and $x=10$, two different Δx spacings corresponding to $\Delta x = 0.25$ and 1 were used. From the results shown in Fig. 1a, we see the rate of convergence with the full Newton (scheme 3, system with five unknowns) is quadratic in all cases and that, with three iterations, the error term in the wall shear parameter f_w'' is reduced from 10^0, 10^{-3} to 10^{-6}. The rate of convergence of the system with four unknowns (scheme 2), on the other hand, is linear. Thus, with both schemes one can specify a small tolerance error ($\equiv 10^{-6}$) and the corresponding results obtained. In contrast, the usual linearization procedure (scheme 1) produces oscillations, restricts the tolerance error that can be specified, and will lead to significantly large computer times if a small convergence criterion is specified.

The computation times of the three linearization schemes, if in each case the calculations are allowed to continue for, say, eight iterations in a given x station, are 0.040, 0.043, and 0.050 min for schemes 1-3, respectively. The computer time associated with scheme 3 is only 20% more than that of scheme 1 because the added Eqs. (15) contribute little to the general A matrix used in the block-elimination method.[5] Of course, the slight increase of computer time associated with scheme 3 can be overcome easily by noting that scheme 3 takes two iterations to achieve a tolerance error of 10^{-3} while scheme 1 takes up to approximately six iterations. This gain increases as the tolerance is decreased.

Figure 1b shows the rate of convergence of the three linearization schemes for a flow with pressure gradient and depicts the same results and conclusions indicated for the zero pressure gradient flow. They show that the rate of convergence is quadratic when Newton's method is applied in full to the momentum equation, and very small values of convergence criterion can be specified without oscillation. The new linearization scheme should lead to substantial savings in computer time in interactive calculations.

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Effect of Angle of Attack on Rotor Trailing-Edge Noise

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Boundary-layer trailing-edge noise has been identified as an important rotor broadband noise mechanism in many cases, especially at high frequencies for large rotors when inflow turbulence is weak.[1] Previous analyses of this noise mechanism used zero blade angle of attack for input data. In practice, to produce desired loadings, rotor blades are operated at various angles of attack. This Note, which is a follow-up to an earlier paper by Kim and George,[2] examines the important effect of change of blade's angle of attack on rotor trailing-edge noise.

Using the same model and assumptions, the general result for the far-field sound pressure level radiated by the turbulent boundary layer passing the rotor blades' trailing edges can be directly adapted from Ref. 2 as

$$\langle S_I(x,f)\rangle = \frac{Bf^2b^2U_c^2\sin^2\phi}{2\pi\rho c_0^3 r^2}\sum_{n=-\infty}^{n=\infty}\frac{F_g(|f-n\Omega|)S_{pp}(|f-n\Omega|)}{(f-n\Omega)^2(1+b/\ell_2(|f-n\Omega|))} \times J_n^2\left(\frac{f}{\Omega}M_0\cos\phi\right) \quad (1)$$

where

b = blade span
B = number of blades
c_0 = the undisturbed sound speed
$c_1 - is_1$ = $E^*[2\mu(1+M)]$
$c_2 - is_2$ = $E^*[2(\mu+\mu M+K_1)]$
f = acoustic frequency, Hz
F_g = F^2+G^2

$$F = \left(\frac{\mu+M\mu+K_1}{\mu+M\mu}\right)^{1/2}\{(c_1+s_1)\cos 2K_1 + (c_1-s_1)\sin 2K_1\}+1-(c_2+s_2)$$

$$G = \left(\frac{\mu+M\mu+K_1}{\mu+M_\mu}\right)^{1/2}\{(c_1-s_1)\cos 2K_1 - (c_1+s_1)\sin 2K_1\}-(c_2-s_2)$$

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